Topic - Integration by Method of substitution

BY - S.K   Date - 30-09-2021

Standard integration

1. (i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1} x/a$ (ii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\{x+\sqrt{x^2+a^2}\}$ (iii) $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ldots$

Form.   Substitution

a^2+x^2 = put $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$a^2+x^2 = a^2\sec^2\theta.$ $\sin\theta = \frac{x}{\sqrt{a^2+x^2}}$ $\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$

$a^2-x^2 \Rightarrow$ Put $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$.

$\Rightarrow \cos\theta = \frac{\sqrt{a^2-x^2}}{a}$ and $\tan\theta = \frac{x}{\sqrt{a^2-x^2}}$

$x^2-a^2 \Rightarrow$ put $x = a\sec\theta. \Rightarrow \sec\theta = \frac{x}{a}$ Also

$\Rightarrow \theta = \sec^{-1} x/a$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$ $\sin\theta = \frac{\sqrt{x^2-a^2}}{a}$

$\frac{a-x}{a+x}$ or $\frac{a+x}{a-x}$ put $x = a\cos 2\theta \Rightarrow \cos 2\theta = x/a$

$a-x = a(1-\cos 2\theta) \Rightarrow \theta = \frac{1}{2}\cos^{-1} x/a$

or $\sqrt{a-x}$ or $\sqrt{a+x}$ $= 2a\sin^2\theta$

$a+x = a(1+\cos 2\theta) = 2a\cos^2\theta.$

Show that-

1. (i) $I = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

put $x = a\tan\theta. \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$dx = a\sec^2\theta\, d\theta \Rightarrow x^2+a^2 = a^2\sec^2\theta.$

$$= \int \frac{a\cancel{\sec^2\theta}\, d\theta}{a^2\cancel{\sec^2\theta}}$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a}$$

(ii) $\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left[x+\sqrt{x^2+a^2}\right]$ show that

Put $x = \cancel{a\cosh}\ a\sinh\theta$ $\quad \because \cosh^2\theta - \sinh^2\theta = 1$

$\theta = \sinh^{-1}\frac{x}{a}$ $\quad \Rightarrow \cosh^2\theta = 1+\sinh^2\theta$

$dx = a\cosh\theta\, d\theta$

$a^2+x^2 = a^2 + a^2\sinh^2\theta$

$= a^2[1+\sinh^2\theta]$

$= a^2\cosh^2\theta.$

$\therefore I = \int \frac{dx}{\sqrt{a^2+x^2}}$

$= \int \frac{a\cosh\theta\, d\theta}{a\cosh\theta}$

$= \int d\theta = \theta = \sinh^{-1}\frac{x}{a} = \log\left[\frac{x+\sqrt{x^2+a^2}}{a}\right]$

$= \log\left[x+\sqrt{x^2+a^2}\right]$ Neglecting $-\log a$

(iii) Show that $\int \sqrt{a^2+x^2}\, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\sinh^{-1}x/a$

$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right]$

$\therefore \sinh^{-1}x/a = \log\frac{x+\sqrt{x^2+a^2}}{a}$

$\cosh^{-1}x/a = \log\frac{x+\sqrt{x^2-a^2}}{a}$

$I = \int \sqrt{a^2+x^2}\, dx$

Put $x = a\sinh\theta$ $\quad a^2+x^2 = a^2\cosh^2\theta$

$dx = a\cosh\theta\, d\theta$ $\quad \cosh^2\theta - \sinh^2\theta = 1$

$\therefore \cosh^2\theta + \sinh^2\theta = \cosh 2\theta$

$\Rightarrow \cosh^2\theta + \cosh^2\theta - 1 = \cosh 2\theta$

$= \int a\,\overset{\cosh\theta}{\cancel{\sinh\theta}}\, a\cosh\theta$ $\quad \Rightarrow \cosh^2\theta = \frac{\cosh 2\theta + 1}{2}$

$= a^2\int \cosh^2\theta.$

$$= \frac{a^2}{2}\int[1+\cosh 2\theta]\,d\theta$$

$$= \frac{a^2}{2}\int d\theta + \frac{a^2}{2}\int \cosh 2\theta\, d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot \frac{\sinh 2\theta}{2}$$

$$= \frac{a^2}{2}\sinh^{-1}x/a + \frac{a^2}{2}\cdot \frac{2\sinh\theta\cosh\theta}{2}$$

$$= \frac{a^2}{2}\sinh^{-1}x/a + \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1+\sinh^2\theta}$$

$$= \frac{a^2}{2}\sinh^{-1}x/a + \frac{a^2}{2a}\frac{x}{a}\sqrt{1+x^2/a^2}$$

$$= \frac{a^2}{2}\sinh^{-1}x/a + \frac{x\sqrt{x^2+a^2}}{2}$$

$$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\frac{x+\sqrt{x^2+a^2}}{a} \qquad \therefore \sinh^{-1}x = \log[x+\sqrt{1+x^2}]$$

$$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right]$$

Neglecting $-\frac{a^2}{2}\log a$

2. Show that (i) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$ $x>a$.

Let $I = \int \frac{dx}{x^2-a^2}$

$$= \int \frac{dx}{(x+a)(x-a)}$$

$$= \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x+a)(x-a)}\,dx$$

$$\therefore I = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right] \qquad \because \int \frac{1}{ax+b}\,dx = \frac{1}{a}\log(ax+$$

$$= \frac{1}{2a}\left[\log(x-a) - \log(x+a)\right]$$

$$= \frac{1}{2a}\log\frac{x-a}{x+a}$$

(ii) Show that-

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left[x+\sqrt{x^2-a^2}\right] = \cosh^{-1}\frac{x}{a} \quad x>a$$

Put- $z = x+\sqrt{x^2-a^2}$

$$\therefore dz = \left[1 + \frac{1}{2\sqrt{x^2-a^2}} \cdot 2x\right]dx$$

$$= \left[\frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}}\right]dx$$

$$= \frac{z}{\sqrt{x^2-a^2}}\,dx$$

$$\Rightarrow \int \frac{dz}{z} = \int \frac{dx}{\sqrt{x^2-a^2}}$$

$$\Rightarrow \log z = \int \frac{dx}{\sqrt{x^2-a^2}}$$

$$\Rightarrow \int \frac{dx}{x^2-a^2} = \log z = \log\left[x+\sqrt{x^2-a^2}\right] = \cosh^{-1}\frac{x}{a}$$

(iii) Show that

$$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left[x+\sqrt{x^2-a^2}\right]$$

Put $x = a\cosh\theta \Rightarrow \theta = \cosh^{-1} x/a$

$dx = a\sinh\theta\, d\theta$

$x^2-a^2 = a^2\sinh^2\theta$

$\sqrt{x^2-a^2} = a\sinh\theta$

$\because$ $\cosh^2\theta - \sinh^2\theta = 1$

$\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$\therefore$ $I = \int a\sinh\theta \cdot a\sinh\theta\, d\theta$

$= a^2 \int \sinh^2\theta \cdot d\theta$

$= \frac{a^2}{2}\int(\cosh 2\theta - 1)\,d\theta$

$= \frac{a^2}{2}\cdot\frac{\sinh 2\theta}{2} - \frac{a^2}{2}\theta$

$= \frac{a^2}{\cancel{2}} \times \frac{\cancel{2}\sinh\theta\cosh\theta}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$

$= \frac{a^2}{2}\,\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1} - \frac{a^2}{2}\cosh^{-1}(x/a)$

$= \frac{a^2}{2}\,\frac{x\sqrt{x^2-a^2}}{a \cdot a} - \frac{a^2}{2}\cosh^{-1} x/a$

$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left[\frac{x+\sqrt{x^2-a^2}}{a}\right]$

Neglecting $-\frac{a^2}{2}\log a$.

Thus $\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right]$

Prove that-

(i) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$

$I = \int \frac{dx}{a^2-x^2}$

$= \int \frac{dx}{(a-x)(a+x)}$

$= \frac{1}{2a}\int \frac{(a+x)+(a-x)}{(a-x)(a+x)} dx$

$= \frac{1}{2a}\left[\int \frac{1}{a-x} dx + \int \frac{1}{a+x} dx\right]$

$= \frac{1}{2a}\left[-\log(a-x) + \log(a+x)\right]$

$= \frac{1}{2a} \log \frac{a+x}{a-x}$

(ii) show that-

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

Put $x = a\sin\theta$ $\Rightarrow \sqrt{a^2-x^2} = a\cos\theta$

$\Rightarrow \theta = \sin^{-1} x/a$

$dx = a\cos\theta\, d\theta$

$I = \int \frac{a\cos\theta\, d\theta}{a\cos\theta}$

$= \int d\theta = \theta = \sin^{-1} x/a$

(iii) $I = \int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

Put $x = a\sin\theta$ $\quad \theta = \sin^{-1}\frac{x}{a}$

$dx = a\cos\theta\, d\theta$

$\sqrt{a^2-x^2} = a\cos\theta$

$= \int a\cos\theta \cdot a\cos\theta\, d\theta$

$= a^2\int \cos^2\theta\, d\theta$

$= a^2\int \frac{1+\cos 2\theta}{2} d\theta$

$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$

$= \frac{a^2}{2}\theta + \frac{a^2}{2} \times \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\,\frac{x}{a}\,\frac{\sqrt{a^2-x^2}}{a}$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2}$

Proved